The lower bound of the Ricci curvature that yields the infinite number of the discrete spectrum of the Laplacian

Hironori Kumura

Abstract

This paper discusses the question whether the discrete spectrum of the Laplace-Beltrami operator is infinite or finite. The borderline-behavior of the curvatures for this problem will be completely determined. Although the topological property of a given manifold M is reflected in that of the cut locus $\mathcal{C}ut(p_0)$ of a point p_0 of M, the main theorem is irrelevant to the property of the cut locus $\mathcal{C}ut(p_0)$. Indeed, it concerns only the Ricci curvatures of the radial direction on $M \backslash \mathcal{C}ut(p_0)$, the complement of the cut locus.

1 Introduction

The Laplace-Beltrami operator Δ on a noncompact complete Riemannian manifold (M, g) is essentially self-adjoit on $C_0^\infty(M)$ and its self-adjoit extension to $L^2(M)$ has been studied by several authors from various points of view. In many cases, the bottom of the essential spectrum of $-\Delta$ will be positive (see Brooks [B]), and the discrete spectrum will appear below this bottom number. The purpose of this paper is to determine the borderline-behavior of curvatures for the question whether the Laplace-Beltrami operator $-\Delta$ has a finite or infinite number of the discrete spectrum. The Rellich's lemma (see, for example, M. Taylor [T]) suggests that this problem depends on the geometry of manifolds *at infinity*. In the case of Schrödinger operators $-\Delta + V$ on the Euclidean space $\mathbf{R}^n$, the borderline-behavior $-\frac{(n-2)^2}{4r^2}$ of the potential V is determined by the *uncertainty principle lemma* $-\Delta \geq \frac{(n-2)^2}{4r^2}$ (see Reed-Simon [R-S] pp. 169 and Kirsh-S [Ki-S]), which is equivalent to the Hardy's inequality $-\frac{d^2u}{dx^2} \geq \frac{1}{4x^2}$ for $u \in C_0^\infty(0, \infty)$ (see, for example, [A-Ku]). Our proof will be concerned with this borderline-behavior of the Hardy's inequality (see Proposition 2.1 in section 2) and use the classical transplantation method adopted by S. Y. Cheng [C].

The main theorem of this paper is the following:

Theorem 1.1. *Let (M, g) be an n-dimensional complete Riemannian manifold and p_0 be a point of M. We set $r(*) := \operatorname{dist}(*, p_0)$ and denote by $\mathcal{C}ut(p_0)$ the cut locus of p_0. Assume that*

$$\min \sigma_{\mathrm{ess}}(-\Delta) = \frac{(n-1)^2\kappa}{4}$$

for some constant $\kappa > 0$ and that there exist positive constants R_0 and β, satisfying $\beta > \frac{1}{(n-1)^2}$, such that

$$\mathrm{Ric}\,(\nabla r, \nabla r) \geq (n-1)\left(-\kappa + \frac{\beta}{r^2}\right) \quad \text{for } x \in M \backslash \mathcal{C}ut(p_0) \text{ with } r(y) \geq R_0,$$

where ∇r stands for the gradient of the function r. Then the set

$$\sigma_{\mathrm{disc}}(-\Delta) \cap \left[0, \frac{(n-1)^2\kappa}{4}\right)$$

is infinite.

Although the topological property of manifolds is reflected in that of the cut locus, the theorem above does *not concern the property of the cut locus at all* but only the *Ricci curvatures of the radial direction on the complement of the cut locus*.

The following proposition shows that the curvature assumption in Theorem 1.1 is sharp:

Proposition 1.1. *Let $\left(\mathbf{R}^n, dr^2 + h^2(r) g_{S^{n-1}(1)}\right)$ be a rotationally symmetric Riemannian manifold and assume that the radial curvature $K(r) = -\frac{h''(r)}{h(r)}$ satisfies*

$$K(r) \leq 0 \qquad \text{for all } \ r \geq 0$$

and there exists constants $\kappa > 0$, $R_0 > 0$ and $\beta \neq \frac{1}{(n-1)^2}$ such that

$$K(r) = -\kappa + \frac{\beta}{r^2} \qquad \text{for } \ r \geq R_0.$$

Then, $\sigma_{\mathrm{ess}}(-\Delta) = \left[\frac{(n-1)^2\kappa}{4}, \infty\right)$, and furthermore, $\sigma_{\mathrm{disc}}(-\Delta) \cap \left[0, \frac{(n-1)^2\kappa}{4}\right)$ is infinite if and only if $\beta > \frac{1}{(n-1)^2}$.

Indeed, under the assumptions in Proposition 1.1, $\mathrm{Ric}\,(\nabla r, \nabla r) = (n-1)K(r) = (n-1)\left(-\kappa + \frac{\beta}{r^2}\right)$, and hence, the lower bound of the Ricci curvature in Theorem 1.1 is sharp. That is, the borderline-behavior of curvatures for our problem can be said to be $-\kappa + \frac{1}{\{(n-1)r\}^2}$. See also [A-Ku] Theorem 3.1 for the finiteness-result on not necessarily rotationally symmetric manifolds.

2 Construction of a model space and eigenfunction

In this section, we shall construct a model space and study the property of an eigenfunction, which will be transplanted on M to prove Theorem 1.1.

Let $R_{\min} : [0, \infty) \to (-\infty, 0]$ be a nonpositive-valued continuous function satisfying

$$\mathrm{Ric}_x(\nabla r, \nabla r) \geq (n-1)R_{\min}(r(x)) \quad \text{for } \ x \in M \backslash \mathcal{C}ut(p_0)$$

and

$$R_{\min}(r) = -\kappa + \frac{\beta}{r^2} \quad \text{for} \quad r \geq R_1, \tag{1}$$

where $\kappa > 0$ and $R_1 > R_0$ are constants.

Using this function $R_{\min}(t)$, consider the solution $J(t)$ to the following classical Jacobi equation:

$$J''(t) + R_{\min}(t)J(t) = 0; \quad J(0) = 0; \quad J'(0) = 1$$

and set

$$S(t) = \frac{J'(t)}{J(t)}.$$

Using this function J, let us consider a model space:

$$M_{\text{model}} := (\mathbf{R}^n, dr^2 + J(r)^2 g_{S^{n-1}(1)}),$$

where r is the Euclidean distance to the origin and $g_{S^{n-1}(1)}$ stands for the standard metric on the unit sphere $S^{n-1}(1)$.

Then, the Laplacian comparison theorem (see Kasue [Ka]) implies that

$$\Delta r = \Delta_{(M,g)}\, r \leq (n-1)S(r). \tag{2}$$

This inequality (2) is known to hold on M in the sense of distribution. Note that $J(t) \geq t > 0$ due to the non-positivity of $R_{\min}$, and hence, $S(t) = \frac{J'(t)}{J(t)}$ exists for all $t \in (0, \infty)$.

Since $S(t) = \frac{J'(t)}{J(t)}$ satisfies the Riccati equation

$$S'(t) + S^2(t) + R_{\min}(t) = 0 \tag{3}$$

and $R_{\min}(t)$ satisfies (1), it is not hard to see that the solution $S(t)$ to this equation (3) has the asymptotic behavior

$$S(t) = \sqrt{\kappa} - \frac{\beta}{2\sqrt{\kappa}\, t^2} + O\left(\frac{1}{t^3}\right). \tag{4}$$

The following proposition serves to construct an eigenfunction on our model space M_{model}:

Proposition 2.1. *For any $R > 0$ and $\delta > 0$, consider the following eigenvalue problem $(*)$:*

$$(*) \begin{cases} -\varphi''(x) - (1+\delta)\dfrac{1}{4x^2}\varphi(x) = \lambda\, \varphi(x) \quad \text{on} \quad [R, 2kR]; \\ \varphi(R) = \varphi(2kR) = 0. \end{cases}$$

Then, the first eigenvalue $-\lambda_1 = -\lambda_1(\delta, R, k)$ of this problem $(*)$ is negative, if $k > 2\left\{\exp\left(\frac{12}{\delta}\right) \wedge 1\right\}$. Here, we write $\exp\left(\frac{12}{\delta}\right) \wedge 1 = \min\left\{\exp\left(\frac{12}{\delta}\right), 1\right\}$.

Proof. We set

$$\chi(x) := \begin{cases} \frac{1}{R}(x - R) & \text{if} \quad x \in [R, 2R], \\ 1 & \text{if} \quad x \in [2R, kR], \\ -\frac{1}{kR}(x - 2kR) & \text{if} \quad x \in [kR, 2kR], \end{cases}$$

where $k > 2$ is a large positive constant defined later. Set $\varphi(x) := \chi(x)x^{\frac{1}{2}}$. Then, the direct computation shows that

$$|\varphi'(x)|^2 - (1+\delta)|\varphi(x)|^2 = |\chi'(x)|^2 x - \frac{\delta}{4x^2}|\varphi(x)|^2 + \frac{1}{2}\left(\chi(x)^2\right)'.$$

Integrating the both sides over $[R, 2kR]$, we have

$$\begin{aligned}
&\int_R^{2kR} \left\{ |\varphi'|^2 - (1+\delta)\frac{1}{4x^2}|\varphi|^2 \right\} dx \\
=&\int_R^{2kR} |\chi'(x)|^2 x\, dx - \frac{\delta}{4}\int_R^{2kR} \frac{\chi^2(x)}{x}\, dx \\
\leq&\frac{1}{R^2}\int_R^{2R} x\, dx + \frac{1}{(kR)^2}\int_{kR}^{2kR} x\, dx - \frac{\delta}{4}\int_{2R}^{kR} \frac{\chi^2(x)}{x}\, dx \\
=&3 - \frac{\delta}{4}\log\left(\frac{k}{2}\right).
\end{aligned}$$

Hence,

$$\int_R^{2kR} \left\{ |\varphi'|^2 - (1+\delta)\frac{1}{4x^2}|\varphi|^2 \right\} dx < 0 \qquad \text{if} \quad k > 2\left\{\exp\left(\frac{12}{\delta}\right) \wedge 1\right\}.$$

Therefore, mini-max principle implies that the first eigenvalue of the problem $(*)$ is negative, if $k > 2\left\{\exp\left(\frac{12}{\delta}\right) \wedge 1\right\}$. □

From our assumption $\beta(n-1)^2 > 1$, we can choose small constant $\delta > 0$ so that

$$\beta(n-1)^2 > 1 + \delta. \tag{5}$$

For a fixed constant $k > 2\left\{\exp\left(\frac{12}{\delta}\right) \wedge 1\right\}$, let $-\lambda_1 = -\lambda_1(k, R, \delta) < 0$ be the first Dirichlet eigenvalue of the problem $(*)$ and $\varphi_1(x)$ be the corresponding eigenfunction. Then, we have

$$\int_R^{2kR} |\varphi_1'(x)|^2\, dx = (1+\delta)\int_R^{2kR} \frac{1}{4x^2}|\varphi_1(x)|^2\, dx - \lambda_1\int_R^{2kR} |\varphi_1(x)|^2\, dx. \tag{6}$$

We set

$$f(x) = \varphi_1(x) J^{-\frac{n-1}{2}}(x).$$

Then, direct computations show that

$$f'(x) = J^{-\frac{n-1}{2}}(x)\left\{ \varphi_1'(x) - \frac{n-1}{2}S(x)\varphi_1(x) \right\}$$

and

$$\begin{aligned}
&|f'(x)|^2 J^{(n-1)}(x) \\
=&|\varphi_1'(x)|^2 + \frac{(n-1)^2}{4}S^2(x)|\varphi_1(x)|^2 - \frac{n-1}{2}S(x)\left\{\varphi_1(x)^2\right\}'.
\end{aligned}$$

As for the last term $-\frac{n-1}{2}S(x)\left\{\varphi_1(x)^2\right\}'$, we calculate

$$-\frac{n-1}{2}\int_R^{2kR} S(x)\left\{\varphi_1(x)^2\right\}'\,dx = \frac{n-1}{2}\int_R^{2kR} S'(x)|\varphi_1(x)|^2\,dx,$$

and hence,

$$\begin{aligned}
&\int_R^{2kR} |f'(x)|^2 J^{n-1}(x)\,dx \\
=&\int_R^{2kR}\left\{|\varphi_1'(x)|^2 + \frac{n-1}{2}\left(\frac{n-1}{2}S^2(x)+S'(x)\right)|\varphi_1(x)|^2\right\}dx \\
=&\int_R^{2kR}\left\{|\varphi_1'(x)|^2 + \frac{n-1}{2}\left(\frac{n-3}{2}S^2(x)-R_{\min}(x)\right)|\varphi_1(x)|^2\right\}dx \\
=&\int_R^{2kR}\left\{\frac{1+\delta}{4x^2}-\lambda_1+\frac{n-1}{2}\left(\frac{n-3}{2}S^2(x)-R_{\min}(x)\right)\right\}|\varphi_1(x)|^2\,dx,
\end{aligned}$$

where we have used equations (3) and (6). Here, by (1) and (4),

$$\begin{aligned}
&\frac{n-1}{2}\left(\frac{n-3}{2}S^2(x)-R_{\min}(x)\right) \\
=&\frac{n-1}{2}\left\{\frac{n-3}{2}\left(\sqrt{\kappa}-\frac{\beta}{2\sqrt{\kappa}\,x^2}+O\left(\frac{1}{x^3}\right)\right)^2+\kappa-\frac{\beta}{x^2}\right\} \\
=&\frac{(n-1)^2\kappa}{4}-\frac{\beta(n-1)^2}{4x^2}+O\left(\frac{1}{x^3}\right)
\end{aligned}$$

and, therefore,

$$\begin{aligned}
&\int_R^{2kR} |f'(x)|^2 J^{n-1}(x)\,dx \\
=&\int_R^{2kR}\left\{\frac{(n-1)^2\kappa}{4}-\lambda_1-\frac{1}{4x^2}\left(\beta(n-1)^2-1-\delta\right)+O\left(\frac{1}{x^3}\right)\right\}|\varphi_1(x)|^2\,dx.
\end{aligned}$$

Since $\beta(n-1)^2-1-\delta>0$ and $|\varphi_1(x)|^2=|f(x)|^2J^{n-1}(x)$, we see that

$$\int_R^{2kR} |f'(x)|^2 J^{n-1}(x)\,dx < \left(\frac{(n-1)^2\kappa}{4}-\lambda_1\right)\int_R^{2kR}|f(x)|^2J^{n-1}(x)\,dx \qquad (7)$$

for $R \geq R_0(n,\beta,\kappa,\delta)$.

Now, for $y\in M_{\mathrm{model}}$, we set

$$\phi(y) := \begin{cases} f(r(y)), & \text{if} \quad r(y)\in[R,2kR], \\ 0, & \text{otherwise.}\end{cases}$$

Then, integrating (7) over $S^{n-1}(1)$ with its standard measure, we have

$$\int_{M_{\mathrm{model}}} |\nabla\phi|^2 dv_{M_{\mathrm{model}}} < \left(\frac{(n-1)^2\kappa}{4}-\lambda_1\right)\int_{M_{\mathrm{model}}}|\phi|^2 dv_{M_{\mathrm{model}}}.$$

We denote by $B(2kR)_{M_{\rm model}}$ the open ball of $M_{\rm model}$ centered at the origin 0 with radius $2kR$, and by $\lambda_D\big(B(2kR)_{M_{\rm model}}\big)$ the first Dirichlet eigenvalue of $-\Delta_{M_{\rm model}}$ on $B(2kR)_{M_{\rm model}}$. Then, mini-max principle implies

$$\lambda_D\big(B(2kR)_{M_{\rm model}}\big) < \frac{(n-1)^2\kappa}{4} - \lambda_1$$

for $R \geq R(n,\beta,\kappa,\delta)$. If we denote by $\widehat{\varphi}_1$ the first Dirichlet eigenfunction of this ball $B(2kR)_{M_{\rm model}}$, $\widehat{\varphi}_1$ is radial, that is,

$$\widehat{\varphi}_1(y) = h_1\left(r(y)\right) \tag{8}$$

for some function $h_1 : [0, 2kR] \to \mathbf{R}$ and h_1 satisfies the equation

$$-h_1''(x) - (n-1)S(x)h_1'(x) = \lambda_D\big(B(2kR)_{M_{\rm model}}\big)h_1(x) \tag{9}$$

on the interval $(0, 2kR]$. Since h_1 takes the same sign on $[0, 2kR)$ (by maximum principle, or see Prüfer [P]), we may assume that

$$h_1(x) > 0 \qquad \text{on} \quad [0, 2kR). \tag{10}$$

Here, we claim the following crucial fact for our proof:

Lemma 2.1. *Under the assumption* (10), h_1 *satisfies*

$$h_1'(x) < 0 \qquad \text{on} \quad (0, 2kR]. \tag{11}$$

Proof. The proof is by contradiction.

First, let us assume that $h_1'(2kR) = 0$. Then, since h_1 satisfies (9) and $h_1(2kR) = 0$, $h_1(x) \equiv 0$ which contradict our assumption (10). Therefore, we see that $h_1'(2kR) < 0$ by (10) and $h_1(2kR) = 0$.

Next, let us assume that $h_1'(x_0) > 0$ for some $x_0 \in (0, 2kR)$. Then, h_1 must takes a minimal value at a point, say x_1, in $[0, x_0)$. If $x_1 \in (0, x_0)$,

$$-h_1''(x_1) = \lambda_D\big(B(2kR)_{M_{\rm model}}\big)h_1(x_1) > 0 \tag{12}$$

by our assumption (10). However, this contradicts our assumption that h_1 takes a minimal value at x_1. Therefore, $x_1 = 0$. Since $h_1'(0) = 0$, $f(0) = 0$, $f'(0) = 1$, and $S(x) = \frac{f'(x)}{f(x)}$, we see that

$$\lim_{x \to +0} S(x)h_1'(x) = h_1''(0),$$

and hence, by (9),

$$-nh_1''(0) = \lambda_D\big(B(2kR)_{M_{\rm model}}\big)h_1(0) > 0. \tag{13}$$

Two equations $h_1'(0) = 0$ and (13) imply that 0 is a maximal point of h_1. However, this contradicts our assertion, proved above, that $x_1 = 0$ is a minimal point of h_1.

Thus, we have proved that

$$h_1'(x) \leq 0 \qquad \text{on} \quad (0, 2kR).$$

However, if $h_1'(x_2) = 0$ for some $x_2 \in (0, 2kR)$, x_2 must be a maximal point of h_1 by the same reason as is seen in (12). Therefore, $h_1'(x_2 - \varepsilon) > 0$ for small $\varepsilon > 0$. This also leads to a contradiction as is seen above. Thus, we have proved (11). $\square$

3 Proof of Theorem 1.1

Let us start with notations involving the cut points of p_0. We set

$$\begin{aligned} U_{p_0}M &= \{v \in T_{p_0}M \mid |v| = 1\}, \\ \mathcal{B}(p_0, \delta) &= \{v \in T_{p_0}M \mid |v| < \delta\} \\ B(p_0, \delta) &= \{y \in M \mid \operatorname{dist}(p_0, y) < \delta\} \end{aligned}$$

and, for each $v \in U_{p_0}M$, define

$$\rho(v) = \sup\left\{t > 0 \mid \operatorname{dist}\left(p_0, \exp_{p_0}(tv)\right) = t\right\}$$

and

$$\mathcal{D}_{p_0} = \{tv \in T_{p_0}M \mid 0 \le t < \rho(v), v \in U_{p_0}M\}.$$

We identify $U_{p_0}M$ with the standard unite sphere $(S^{n-1}(1), g_0)$ and write the Riemannian measure dv_g on the domain $\exp_{p_0}(\mathcal{D}_{p_0})$ as follows:

$$dv_g = \sqrt{g}(r, v)\, dr\, d\mu_{n-1}, \tag{14}$$

where $r = \operatorname{dist}(p_0, *)$ and $d\mu_{n-1}$ is the Riemannian measure on the $(n-1)$-dimensional standard unit sphere $U_{p_0}M$.

As in S. y. Cheng [C], we use the transplantation method.

For $(t, v) \in [0, \infty) \times U_{p_0}M$ satisfying $tv \in \overline{\mathcal{B}(p_0, R)} \cap \overline{\mathcal{D}_{p_0}}$, define $F_R = F$ by

$$F\left(\exp_{p_0}(tv)\right) = F_R\left(\exp_{p_0}(tv)\right) = h_1(t),$$

where h_1 is the function defined by (8). Then $F \in W_c^{1,2}(B(p_0, R))$ and

$$\begin{aligned} \int_{B(p_0,R)} |\nabla F|^2 dv_g &= \int_{U_{p_0}M} d\mu_{n-1}(v) \int_0^{\rho(v)\wedge R} |h_1'|^2 \sqrt{g}(r, v)\, dr\,; \\ \int_{B(p_0,R)} |F|^2 dv_g &= \int_{U_{p_0}M} d\mu_{n-1}(v) \int_0^{\rho(v)\wedge R} |h_1|^2 \sqrt{g}(r, v)\, dr. \end{aligned}$$

Now, for each $v \in U_{p_0}M$,

$$\begin{aligned} &\int_0^{\rho(v)\wedge R} |h_1'|^2 \sqrt{g}(r, v)\, dr \\ =&\Big[h_1 h_1' \sqrt{g}(r, v)\Big]_0^{\rho(v)\wedge R} - \int_0^{\rho(v)\wedge R} h_1 \big\{h_1' \sqrt{g}(r, v)\big\}' dr \\ =&(h_1 h_1')\,(\rho(v) \wedge R) \cdot \sqrt{g}\,(\rho(v) \wedge R, v) - \int_0^{\rho(v)\wedge R} h_1 \big\{h_1' \sqrt{g}(r, v)\big\}' dr \\ \le& -\int_0^{\rho(v)\wedge R} h_1 \big\{h_1' \sqrt{g}(r, v)\big\}' dr \\ =& -\int_0^{\rho(v)\wedge R} h_1 \left\{h_1'' + \frac{\partial_r \sqrt{g}(r, v)}{\sqrt{g}(r, v)} h_1'\right\} \sqrt{g}(r, v)\, dr \\ \le& -\int_0^{\rho(v)\wedge R} h_1 \left\{h_1'' + (n-1)S(r)h_1'\right\} \sqrt{g}(r, v)\, dr \\ =&\lambda_D\big(B(2kR)_{M_{\mathrm{model}}}\big) \int_0^{\rho(v)\wedge R} |h_1|^2 \sqrt{g}(r, v)\, dr, \end{aligned}$$

where we have used (10) and (11) at the first inequality, and (10), (11), $\Delta r = \frac{\partial_r \sqrt{g}}{\sqrt{g}}$, and (2) at the second inequality, and (9) at the last line.

Therefore, integrating both side of this inequality over $U_{p_0}M$, we see that

$$\begin{aligned}\frac{\int_{B(p_0,R)} |\nabla F|^2 dv_g}{\int_{B(p_0,R)} |F|^2 dv_g} \leq & \lambda_D\big(B(2kR)_{M_{\mathrm{model}}}\big) \\ < & \frac{(n-1)^2\kappa}{4} - \lambda_1(R,\delta,\kappa).\end{aligned}$$

This inequality holds for all $R \geq R_0(n,\beta,\kappa,\delta)$, and hence, setting $R_i = R_0(n,\beta,\kappa,\delta)+i$ and considering the corresponding functions F_{R_i} as above, we get the sequence $\{F_{R_i}\}$ of functions in $W_c^{1,2}(M)$ satisfying

$$\begin{aligned}&\frac{\int_{B(p_0,R)} |\nabla F_{R_i}|^2 dv_g}{\int_{B(p_0,R)} |F_{R_i}|^2 dv_g} < \frac{(n-1)^2\kappa}{4}; \\ &\operatorname{supp} F_{R_i} = \overline{B(p_0,R_i)}.\end{aligned}$$

Since $\{F_{R_i}\}_{i=1}^{\infty}$ spans the infinite dimensional subspace in $W_c^{1,2}(M)$, we obtain the conclusion of Theorem 1.1 by mini-max principle.

The proof of Proposition 1.1

Proposition 1.1 easily follows from the asymptotic behavior of the shape operator of the level hypersurfaces of r

$$\frac{h'(r)}{h(r)} = \sqrt{\kappa} - \frac{\beta}{2\sqrt{\kappa}\, r^2} + O\left(\frac{1}{r^3}\right),$$

and Theorem 1.1 in [A-Ku], and discussions there. For details, see that paper.

References

[A-Ku] K. Akutagawa and H. Kumura, *The uncertainty principle lemma under gravity and the discrete spectrum of Schrödinger operators*, arXiv:0812.4663.

[B] R. Brooks, A relation between growth and the spectrum of the Laplacian, Math. Z., **178** (1981), 501–508.

[C] S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z, **143** (1982), 289–297.

[Ka] A. Kasue, *Applications of Laplacian and Hessian comparison theorems*, Geometry of geodesics and related topics (Tokyo, 1982), 333–386, Adv. Stud. Pure Math., 3.

[Ki-S] W. Kirsch and B. Simon, *Corrections to the classical behavior of the number of bound states of Schrödinger operators*, Ann. Phys., **183** (1988), 122–130.

[P] H. Prüfer, *Neue Herleitung der Sturm-Liouvilleschen Reihenentwicklung stetiger Funktionen*, Math. Ann., **95** (1926), 499–518.

[R-S] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. II, Academic Press, 1972.

[T] M. E. Taylor, Partial Differential Equations I, (Applied Math. Sci. 116), Springer-Verlag, 1996.

Hironori Kumura
Department of Mathematics
Shizuoka University
Ohya, Shizuoka 422-8529
Japan
E-mail address: smhkumu@ipc.shizuoka.ac.jp